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Bounds for complete elliptic integrals of the first kind

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ABSTRACT

In this note by using some elementary computations we present some new sharp lower and upper bounds for the complete elliptic integrals of the first kind. These results improve some known bounds in the literature and are deduced from the well-known Wallis inequality, which has been studied extensively in the last 10 years.

1. Bounds for the Legendre complete elliptic integral of the first kind

Let us consider the sequence $\{w_n\}_{n\geq 1}$, defined for all $n \geq 1$ integer by the relation

$$w_n := \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(n+1)} = \frac{(2n-1)!!}{(2n)!!},$$

where, as usual, Γ stands for the Euler gamma function. This sequence is related to the well-known Wallis' product formula of approximation of π and it has been studied by many mathematicians in the last 10 years. The interested reader is referred to the papers [5,6] and to the references therein. In

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order to improve Kazarinoff's result [7, p. 192], i.e.

$$\frac{1}{\sqrt{\pi(n+1/2)}} < \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} < \frac{1}{\sqrt{\pi(n+1/4)}},$$

recently, Chen and Qi [5] proved the following sharp inequalities for the sequence $\{w_n\}_{n\geq 1}$

$$\frac{1}{\sqrt{\pi(n+\mu_1)}} \leq \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} < \frac{1}{\sqrt{\pi(n+\mu_2)}}, \tag{1}$$

which hold for all $n\geq 1$, and the constants $\mu_1 = 4/\pi - 1$ and $\mu_2 = 1/4$ are the best possible. Even if these inequalities are interesting in their own right, in this section we show that (1) can be used to deduce sharp lower and upper bounds for the Legendre elliptic integral of the first kind. Not surprisingly, these bounds can be represented as hypergeometric functions and for the reader's convenience we recall some basic facts.

Let ${}_2F_1(a,b,c,r)$ denote the Gaussian hypergeometric series (function), which for real numbers a,b,c and $c\neq 0,-1,-2,\ldots$ has the infinite series representation

$${}_2F_1(a,b,c,r) := \sum_{n\geq 0} \frac{(a)_n(b)_n}{(c)_n}\cdot\frac{r^n}{n!} \quad \text{for all } r\in(-1,1), \tag{2}$$

where $(a)_0 = 1$ for $a\neq 0$ and $(a)_n = a(a+1)\ldots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ for each $n = 1,2,\ldots$ denotes the well-known Pochhammer (or Appell) symbol. A special case of the zero-balanced Gaussian hypergeometric functions ${}_2F_1(a,b,a+b,r)$, namely the Legendre complete elliptic integral of the first kind $\mathcal{K}$ is of particular interest in many problems of physics, engineering, geometry, geometric function theory, quasiconformal analysis, theory of mean values and number theory. This elliptic integral is defined as follows

$$\mathcal{K}(r) := \frac{\pi}{2}\cdot {}_2F_1\left(\frac{1}{2},\frac{1}{2},1,r^2\right) = \int_0^{\pi/2}(1-r^2\sin^2 t)^{-1/2}\,\mathrm{d}t,$$

where $r\in(0,1)$. Motivated by the importance of this elliptic integral many authors deduced interesting symmetric lower and upper bounds in terms of elementary functions. For more details about this special function, monotonicity properties, symmetric lower and upper bounds the interested reader is referred to the papers [1–4] and to the references therein. Since from (2) the Legendre complete elliptic integral of the first kind can be rewritten as follows

$$\mathcal{K}(r) = \frac{\pi}{2}\sum_{n\geq 0}\frac{(1/2)_n(1/2)_n}{(1)_n n!}r^{2n} = \frac{\pi}{2}\sum_{n\geq 0}\left[\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)}\right]^2 r^{2n},$$

the following result is an immediate consequence of (1).

Theorem 1. *Let us consider the functions $f,g:(0,1)\to\mathbb{R}$, defined by*

$$f(r) := \mathcal{K}(r) - \frac{\pi}{2} - \frac{1}{2\mu_1}[{}_2F_1(1,\mu_1,\mu_1+1,r^2)-1],$$
$$g(r) := \mathcal{K}(r) - \frac{\pi}{2} - \frac{1}{2\mu_2}[{}_2F_1(1,\mu_2,\mu_2+1,r^2)-1],$$

where $\mu_1 = 4/\pi - 1$ and $\mu_2 = 1/4$. Then, the MacLaurin series of the functions f and $-g$ have constant term zero and all other coefficients positive. In particular, we have the following lower and upper bounds for the Legendre complete elliptic integral of the first kind:

$$\frac{\pi}{2} + \frac{1}{2\mu_1}[{}_2F_1(1,\mu_1,\mu_1+1,r^2)-1] < \mathcal{K}(r) < \frac{\pi}{2} + \frac{1}{2\mu_2}[{}_2F_1(1,\mu_2,\mu_2+1,r^2)-1], \tag{3}$$

where $r\in(0,1)$ and both inequalities are sharp as r tends to zero.

Proof. Elementary computations show that

$$\begin{aligned} f(r) &= \mathcal{K}(r) - \frac{\pi}{2} - \frac{1}{2\mu_1}[{}_2F_1(1,\mu_1,\mu_1+1,r^2)-1] \\ &= \frac{\pi}{2}\left[{}_2F_1\left(\frac{1}{2},\frac{1}{2},1,r^2\right)-1\right] - \frac{1}{2\mu_1}[{}_2F_1(1,\mu_1,\mu_1+1,r^2)-1] \\ &= \frac{\pi}{2}\sum_{n\ge1}\frac{(1/2)_n(1/2)_n}{(1)_n n!}r^{2n} - \frac{1}{2\mu_1}\sum_{n\ge1}\frac{(1)_n(\mu_1)_n}{(\mu_1+1)_n n!}r^{2n} \\ &= \frac{\pi}{2}\sum_{n\ge1}\frac{\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2}{\pi(n!)^2}r^{2n} - \frac{\pi}{2}\sum_{n\ge1}\frac{1}{\pi(n+\mu_1)}r^{2n} \\ &= \frac{\pi}{2}\sum_{n\ge1}\left[\left[\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(n+1)}\right]^2 - \frac{1}{\pi(n+\mu_1)}\right]r^{2n} > 0, \end{aligned}$$

and

$$\begin{aligned} g(r) &= \mathcal{K}(r) - \frac{\pi}{2} - \frac{1}{2\mu_2}[{}_2F_1(1,\mu_2,\mu_2+1,r^2)-1] \\ &= \frac{\pi}{2}\left[{}_2F_1\left(\frac{1}{2},\frac{1}{2},1,r^2\right)-1\right] - \frac{1}{2\mu_2}[{}_2F_1(1,\mu_2,\mu_2+1,r^2)-1] \\ &= \frac{\pi}{2}\sum_{n\ge1}\frac{(1/2)_n(1/2)_n}{(1)_n n!}r^{2n} - \frac{1}{2\mu_2}\sum_{n\ge1}\frac{(1)_n(\mu_2)_n}{(\mu_2+1)_n n!}r^{2n} \\ &= \frac{\pi}{2}\sum_{n\ge1}\frac{\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2}{\pi(n!)^2}r^{2n} - \frac{\pi}{2}\sum_{n\ge1}\frac{1}{\pi(n+\mu_2)}r^{2n} \\ &= \frac{\pi}{2}\sum_{n\ge1}\left[\left[\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(n+1)}\right]^2 - \frac{1}{\pi(n+\mu_2)}\right]r^{2n} < 0, \end{aligned}$$

where in view of (1) we have used the sharp inequalities

$$\frac{1}{\pi(n+\mu_1)} \le \left[\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(n+1)}\right]^2 < \frac{1}{\pi(n+\mu_2)}, \tag{4}$$

which hold for all $n \ge 1$, and the constants $\mu_1 = 4/\pi - 1$ and $\mu_2 = 1/4$ are the best possible. □

2. Bounds for the generalized complete elliptic integrals of the first kind

Recently, Koumandos [6] pointed out that the improved Wallis inequality (1) in fact follows easily from a well-known result of Watson. Moreover, he pointed out that it can be shown that for each $a \in (0,1)$ and $n \ge 1$

$$\frac{1}{\Gamma(1-a)(n+1)^a} < \frac{(1-a)_n}{n!} < \frac{1}{\Gamma(1-a)n^a}. \tag{5}$$

Changing in the above inequality a with $1-a$ we obtain that

$$\frac{1}{\Gamma(a)(n+1)^{1-a}} < \frac{(a)_n}{n!} < \frac{1}{\Gamma(a)n^{1-a}}$$

holds too for all $a \in (0,1)$ and $n \ge 1$. Now multiplying the corresponding parts of these inequalities and using the reflection property of the gamma function, i.e.

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin\pi a},$$

we obtain

$$\frac{\sin\pi a}{\pi}\frac{1}{n+1} < \frac{(a)_n(1-a)_n}{(1)_n n!} < \frac{\sin\pi a}{\pi}\frac{1}{n}. \tag{6}$$

Summing up these inequalities for $n \geq 1$ it can be shown that the generalized elliptic integral of the first kind $\mathcal{K}_a$, defined by

$$\mathcal{K}_a(r) := \frac{\pi}{2} \cdot {}_2F_1(a, 1-a, 1, r^2),$$

where $a, r \in (0,1)$, satisfies the following inequalities

$$\frac{\pi}{2}\left[1 - \frac{\sin \pi a}{\pi}\left[\frac{1}{r^2}\log(1-r^2)+1\right]\right] < \mathcal{K}_a(r) < \frac{\pi}{2}\left[1 - \frac{\sin \pi a}{\pi}\log(1-r^2)\right], \tag{7}$$

which holds for all $a, r \in (0,1)$ and both of inequalities are sharp as r tends to zero. However, due to Koumandos [6] there is an improvement of (5), which we will use as follows to deduce more tight bounds to those presented in (7) for the generalized elliptic integral of the first kind $\mathcal{K}_a$. To achieve our goal first we improve (6).

Lemma 1. *For all $a \in (0,1)$ and $n \geq 1$ integer the following sharp inequalities hold*

$$\frac{\sin \pi a}{\pi}\frac{1}{n+\mu_1(a)} < \frac{(a)_n(1-a)_n}{(1)_n n!} < \frac{\sin \pi a}{\pi}\frac{1}{n+\mu_2(a)}, \tag{8}$$

where the constants

$$\mu_1(a) := \frac{a}{[\Gamma(2-a)]^{1/a}} + \frac{1-a}{[\Gamma(1+a)]^{1/(1-a)}} - 1, \mu_2(a) := \left(1+\frac{a}{2}\right)\left(\frac{3-a}{2+a}\right)^a - 1$$

are the best possible.

Proof. Recently, in order to improve (5), Koumandos [6] established the following generalization of the improved Wallis' inequality (1)

$$\frac{1}{\Gamma(1-a)[n+\nu_1(a)]^a} \leq \frac{(1-a)_n}{n!} < \frac{1}{\Gamma(1-a)[n+\nu_2(a)]^a}, \tag{9}$$

where $a \in (0,1)$, $n \geq 1$ and the constants

$$\nu_1(a) = \frac{1}{[\Gamma(2-a)]^{1/a}} - 1 \quad \text{and} \quad \nu_2(a) = \frac{1-a}{2}$$

are the best possible. Using (9) and the reflection property of the gamma function we have

$$\frac{\sin \pi a}{\pi}\frac{1}{[n+\nu_1(a)]^a[n+\nu_1(1-a)]^{1-a}} \leq \frac{(a)_n(1-a)_n}{(1)_n n!}$$

and

$$\frac{(a)_n(1-a)_n}{(1)_n n!} < \frac{\sin \pi a}{\pi}\frac{1}{[n+\nu_2(a)]^a[n+\nu_2(1-a)]^{1-a}},$$

where $a \in (0,1)$ and $n \geq 1$. In order to prove (8) in what follows we show that for each $a \in (0,1)$ and $n \geq 1$ the following inequalities hold true

$$\frac{1}{[n+\nu_2(a)]^a[n+\nu_2(1-a)]^{1-a}} \leq \frac{1}{n+\mu_2(a)}, \tag{10}$$

$$\frac{1}{[n+\nu_1(a)]^a[n+\nu_1(1-a)]^{1-a}} > \frac{1}{n+\mu_1(a)}, \tag{11}$$

and in these inequalities the constants $\mu_1(a)$ and $\mu_2(a)$ are the best possible. First observe that (10) is equivalent to

$$\mu_2(a) \leq [n+\nu_2(1-a)]\left[\frac{n+\nu_2(a)}{n+\nu_2(1-a)}\right]^a - n, \tag{12}$$

and (11) is equivalent to

$$\mu_1(a) > [n+v_1(1-a)]\left[\frac{n+v_1(a)}{n+v_1(1-a)}\right]^a - n. \tag{13}$$

Now, consider the function $f_{\alpha,\beta}:[0,\infty)\to\mathbb{R}$, defined by

$$f_{\alpha,\beta}(x) := (x+\alpha)\left(\frac{x+\beta}{x+\alpha}\right)^a - x,$$

where $a\in(0,1)$ and $\alpha,\beta>0$, $\alpha\neq\beta$. Then we have

$$f_{\alpha,\beta}'(x) = \left(\frac{x+\beta}{x+\alpha}\right)^a + a\left(\frac{x+\beta}{x+\alpha}\right)^{a-1}\frac{\alpha-\beta}{x+\alpha} - 1,$$

and this is positive if and only if for all $x\geq 0$, $a\in(0,1)$ and $\alpha,\beta>0$, $\alpha\neq\beta$ we have

$$\left(\frac{x+\alpha}{x+\beta}\right)^a < 1 + a\cdot\frac{\alpha-\beta}{x+\beta}.$$

But, the well-known Bernoulli inequality [7, p. 34] states that if $a\in(0,1)$ and $y>-1$, $y\neq 0$, then we have

$$(y+1)^a < 1+ay.$$

Choosing in Bernoulli's inequality $y=(\alpha-\beta)/(x+\beta)$ we conclude that $f_{\alpha,\beta}'(x)>0$ for all $x\geq 0$ and $\alpha,\beta>0$, $\alpha\neq\beta$, i.e. the function $f_{\alpha,\beta}$ is strictly increasing. Consequently we obtain that for each $a\in(0,1)$ and $n\geq 1$ the inequality $f_{v_2(1-a),v_2(a)}(n)\geq f_{v_2(1-a),v_2(a)}(1)=\mu_2(a)$, i.e. (12) holds true. This implies that (10) holds true and thus the proof of the upper bound in (8) is done. On the other hand it is easy to see that $\lim_{x\to\infty} f_{\alpha,\beta}(x) = a\beta+(1-a)\alpha$. Thus using again the fact that $f_{\alpha,\beta}$ is strictly increasing we obtain that

$$f_{v_1(1-a),v_1(a)}(n) < \lim_{n\to\infty} f_{v_1(1-a),v_1(a)}(n) = av_1(a)+(1-a)v_1(1-a) = \mu_1(a),$$

i.e. (13), and in addition (11) holds. This completes the proof. $\square$

We are now in a position to improve (7) and to generalize (3).

Theorem 2. *Let us consider the functions $f_a, g_a:(0,1)\to\mathbb{R}$, defined by*

$$f_a(r) := \mathcal{K}_a(r) - \frac{\pi}{2} - \frac{\sin\pi a}{2\mu_1(a)}[{}_2F_1(1,\mu_1(a),\mu_1(a)+1,r^2)-1],$$
$$g_a(r) := \mathcal{K}_a(r) - \frac{\pi}{2} - \frac{\sin\pi a}{2\mu_2(a)}[{}_2F_1(1,\mu_2(a),\mu_2(a)+1,r^2)-1],$$

where $a\in(0,1)$ and $\mu_1(a)$, $\mu_2(a)$ are as in Lemma 1. Then, the MacLaurin series of the functions f_a and $-g_a$ have constant term zero and all other coefficients positive. In particular, we have the following lower and upper bounds for the generalized complete elliptic integral of the first kind:

$$\frac{\pi}{2} + \frac{\sin\pi a}{2\mu_1(a)}[{}_2F_1(1,\mu_1(a),\mu_1(a)+1,r^2)-1] < \mathcal{K}_a(r), \tag{14}$$

$$\mathcal{K}_a(r) < \frac{\pi}{2} + \frac{\sin\pi a}{2\mu_2(a)}[{}_2F_1(1,\mu_2(a),\mu_2(a)+1,r^2)-1], \tag{15}$$

where $a,r\in(0,1)$ and both inequalities are sharp as r tends to zero.

Proof. Our strategy is as in the proof of Theorem 1. Elementary computations show that

$$\begin{aligned} f_a(r) &= \mathcal{K}_a(r) - \frac{\pi}{2} - \frac{\sin\pi a}{2\mu_1(a)}[{}_2F_1(1,\mu_1(a),\mu_1(a)+1,r^2)-1] \\ &= \frac{\pi}{2}[{}_2F_1(a,1-a,1,r^2)-1] - \frac{\sin\pi a}{2\mu_1(a)}[{}_2F_1(1,\mu_1(a),\mu_1(a)+1,r^2)-1] \end{aligned}$$

$$= \frac{\pi}{2}\sum_{n\geq 1}\frac{(a)_n(1-a)_n}{(1)_n n!}r^{2n} - \frac{\sin\pi a}{2\mu_l(a)}\sum_{n\geq 1}\frac{(1)_n(\mu_1(a))_n}{(\mu_1(a)+1)_n n!}r^{2n}$$
$$= \frac{\pi}{2}\sum_{n\geq 1}\frac{(a)_n(1-a)_n}{(1)_n n!}r^{2n} - \frac{\pi}{2}\sum_{n\geq 1}\frac{\sin\pi a}{\pi}\frac{1}{n+\mu_1(a)}r^{2n}$$
$$= \frac{\pi}{2}\sum_{n\geq 1}\left[\frac{(a)_n(1-a)_n}{(1)_n n!} - \frac{\sin\pi a}{\pi}\frac{1}{n+\mu_1(a)}\right]r^{2n} > 0,$$

and

$$g_a(r) = \mathcal{K}_a(r) - \frac{\pi}{2} - \frac{\sin\pi a}{2\mu_2(a)}[{}_2F_1(1,\mu_2(a),\mu_2(a)+1,r^2)-1]$$
$$= \frac{\pi}{2}[{}_2F_1(a,1-a,1,r^2)-1] - \frac{\sin\pi a}{2\mu_2(a)}[{}_2F_1(1,\mu_2(a),\mu_2(a)+1,r^2)-1]$$
$$= \frac{\pi}{2}\sum_{n\geq 1}\frac{(a)_n(1-a)_n}{(1)_n n!}r^{2n} - \frac{\sin\pi a}{2\mu_2(a)}\sum_{n\geq 1}\frac{(1)_n(\mu_2(a))_n}{(\mu_u(a)+1)_n n!}r^{2n}$$
$$= \frac{\pi}{2}\sum_{n\geq 1}\frac{(a)_n(1-a)_n}{(1)_n n!}r^{2n} - \frac{\pi}{2}\sum_{n\geq 1}\frac{\sin\pi a}{\pi}\frac{1}{n+\mu_2(a)}r^{2n}$$
$$= \frac{\pi}{2}\sum_{n\geq 1}\left[\frac{(a)_n(1-a)_n}{(1)_n n!} - \frac{\sin\pi a}{\pi}\frac{1}{n+\mu_2(a)}\right]r^{2n} < 0,$$

where we have used the sharp inequality (8) from Lemma 1. □

3. Concluding remarks

1. We note that since $\mathcal{K}_{1/2} = \mathcal{K}$, $\mu_1(1/2) = \mu_1$, $\mu_2(1/2) = \mu_2$, $f_{1/2} = f$ and $g_{1/2} = g$ it is clear that for $a = 1/2$, inequality (14) coincides with the left hand side of (3), and inequality (15) coincides with the right hand side of (3). Thus Theorem 2 is a natural generalization of Theorem 1.
2. In 1992 Anderson et al. [3] proved that the Legendre complete elliptic integral of the first kind $\mathcal{K}_{1/2} = \mathcal{K}$ can be approximated by the inverse hyperbolic tangent function arth, i.e.

   $$\operatorname{arth} r = \frac{1}{2}\log\left(\frac{1+r}{1-r}\right) = r\cdot {}_2F_1\left(\frac{1}{2},1,\frac{3}{2},r^2\right), \text{ where } r\in(0,1).$$

   More precisely, Anderson et al. proved that for $r\in(0,1)$ we have

   $$\frac{\pi}{2}\cdot\left(\frac{\operatorname{arth} r}{r}\right)^{1/2} < \mathcal{K}(r) < \frac{\pi}{2}\cdot\frac{\operatorname{arth} r}{r}. \tag{16}$$

   It is also worth mentioning that, recently, the left hand side of inequality (16) was improved by Alzer and Qiu [1]. They proved that for all $r\in(0,1)$ we have

   $$\frac{\pi}{2}\cdot\left(\frac{\operatorname{arth} r}{r}\right)^{\alpha} < \mathcal{K}(r) < \frac{\pi}{2}\cdot\left(\frac{\operatorname{arth} r}{r}\right)^{\beta} \tag{17}$$

   with the best possible constants $\alpha = 3/4$ and $\beta = 1$. We note that our upper bound from (3) is tighter than the upper bound from (16), or (17). To see this, consider the function $h:(0,1)\to\mathbb{R}$, defined by the relation

   $$h(r) := \frac{\pi}{2}\cdot\frac{\operatorname{arth} r}{r} - \frac{\pi}{2} - \frac{1}{2\mu_2}[{}_2F_1(1,\mu_2,\mu_2+1,r^2)-1].$$

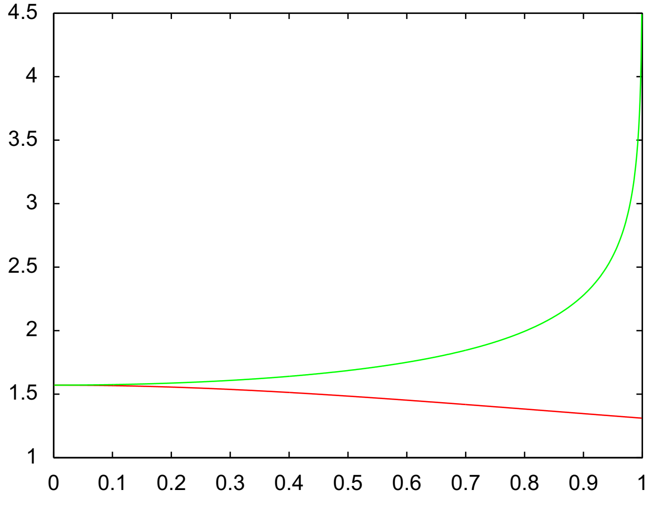

Fig. 1. The graph of the functions u and v.

Then some elementary computations yields

$$\begin{aligned} h(r) &= \frac{\pi}{2}\left[{}_2F_1\left(\frac{1}{2},1,\frac{3}{2},r^2\right)-1\right]-2\left[{}_2F_1\left(1,\frac{1}{4},\frac{5}{4},r^2\right)-1\right] \\ &= \frac{\pi}{2}\sum_{n\geq 1}\frac{(1/2)_n(1)_n}{(3/2)_n}\frac{r^{2n}}{n!}-2\sum_{n\geq 1}\frac{(1)_n(1/4)_n}{(5/4)_n}\frac{r^{2n}}{n!} \\ &= \sum_{n\geq 1}\left[\frac{\pi}{2}\frac{(1/2)_n}{(3/2)_n}-2\frac{(1/4)_n}{(5/4)_n}\right]r^{2n} \\ &= \sum_{n\geq 1}\left[\frac{\pi}{2}\frac{1}{2n+1}-\frac{2}{4n+1}\right]r^{2n}. \end{aligned}$$

Since

$$n\geq 1>\frac{4-\pi}{4\pi-8},$$

it follows that the MacLaurin series of the function h has constant term zero and all other coefficients positive. In particular, for each $r\in(0,1)$ we have $h(r)>0$, and this implies that the right hand side of (16) is weaker than the right hand side of (3).

3. Moreover, based on numerical experiments, we claim that our lower bound from (3) is tighter than the lower bound from (17) (see Fig. 1). Thus, we end this paper with the following conjecture: **Conjecture**. For all $r\in(0,1)$ we have

$$u(r)<v(r),$$

where

$$u(r)=\frac{\pi}{2}\cdot\left(\frac{\operatorname{arth} r}{r}\right)^{3/4}$$

and

$$v(r)=\frac{\pi}{2}+\frac{1}{2(4/\pi-1)}[{}_2F_1(1,4/\pi-1,4/\pi,r^2)-1].$$

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